# ESTIMATES FOR THE ASYMPTOTIC BEHAVIOR OF THE CONSTANTS IN THE BOHNENBLUST–HILLE INEQUALITY

G. A. MUÑOZ-FERNÁNDEZ*, D. PELLEGRINO**, AND J. B. SEOANE-SEPÚLVEDA*

Abstract. A classical inequality due to H.F. Bohnenblust and E. Hille states that for every positive integer $n$ there is a constant $C_n > 0$ so that

$$\left( \sum_{i_1,\ldots,i_n=1}^{N} \left| U(e_{i_1},\ldots,e_{i_n}) \right|^{\frac{2n}{n+1}} \right)^{\frac{n+1}{2n}} \leq C_n \|U\|$$

for every positive integer $N$ and every $n$-linear mapping $U : \ell_\infty^N \times \cdots \times \ell_\infty^N \to \mathbb{C}$. The original estimates for those constants from Bohnenblust and Hille are

$$C_n = n^{\frac{n+1}{2n}} 2^{\frac{n-1}{2}}.$$

In this note we present explicit formulae for quite better constants, and calculate the asymptotic behavior of these estimates, completing recent results of the second and third authors. For example, we show that, if $C_{\mathbb{R},n}$ and $C_{\mathbb{C},n}$ denote (respectively) these estimates for the real and complex Bohnenblust–Hille inequality then, for every even positive integer $n$,

$$\frac{C_{\mathbb{R},n}}{\sqrt{\pi}} = \frac{C_{\mathbb{C},n}}{\sqrt{2}} = 2^{\frac{n+2}{8}} \cdot r_n$$

for a certain sequence $\{r_n\}$ which we estimate numerically to belong to the interval $(1, 3/2)$ (the case $n$ odd is similar). Simultaneously, assuming that $\{r_n\}$ is in fact convergent, we also conclude that

$$\lim_{n\to\infty} \frac{C_{\mathbb{R},n}}{C_{\mathbb{R},n-1}} = \lim_{n\to\infty} \frac{C_{\mathbb{C},n}}{C_{\mathbb{C},n-1}} = 2^{\frac{1}{8}}.$$



## 1. Preliminaries and background

Since Lindenstrauss and Pełczyński's classical paper [15], the theory of absolutely summing linear operators became an important topic of research in Functional Analysis (see [10] and references therein). The most famous constant involved in the theory of absolutely linear operators is the constant from Grothendieck's fundamental theorem in the metric theory of tensor products, called Grothendieck's constant $K_G$. In recent years, Grothendieck's type inequalities have received a significant amount of attention in view of their various applications (see, e. g., [1,11]). Grothendieck's famous Resumé asks for determining the precise value of $K_G$ (see [12, Problem 3], [9] and references therein). However, this problem remains open despite of the progress made. For instance, it is well-known that for real scalars

$$K_G \leq \frac{\pi}{2\log\left(1+\sqrt{2}\right)}.$$

For some time it was believed that, in fact, this inequality was sharp, but not until very recently [2] it was proved that it is actually not.

In the multilinear theory of absolutely summing operators the key constants are the constants $C_n$ involved in the Bohnenblust–Hille inequality, which we describe below. In 1930 J.E. Littlewood proved that

$$\left( \sum_{i,j=1}^{N} |U(e_i,e_j)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \sqrt{2}\,\|U\|$$

for every bilinear form $U : \ell_\infty^N \times \ell_\infty^N \to \mathbb{C}$ and every positive integer $N$. This is the well-known Littlewood's 4/3 inequality [16]. Just one year later, H.F. Bohnenblust and E. Hille observed

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that Littlewood's inequality had important connection with Bohr's absolute convergence problem which consists in determining the maximal width $T$ of the vertical strip in which a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges uniformly but not absolutely. Bohnenblust and Hille proved that $T = 1/2$ and for this task they improved Littlewood's 4/3 by showing that for all positive integer $n > 2$ there is a $C_n > 0$ so that

$$\left( \sum_{i_1,\dots,i_n=1}^{N} \left| U(e_{i_1}, \dots, e_{i_n}) \right|^{\frac{2n}{n+1}} \right)^{\frac{n+1}{2n}} \leq C_n \|U\| \tag{1.1}$$

for all $n$-linear mapping $U : \ell_\infty^N \times \cdots \times \ell_\infty^N \to \mathbb{C}$ and every positive integer $N$. In their paper Bohnenblust and Hille showed that $C_n = n^{\frac{n+1}{2n}} 2^{\frac{n-1}{2}}$ is a valid constant (for related recent papers we refer the reader to [6, 7]).

This inequality was overlooked for a long time and rediscovered later by A. Davie [5] and S. Kaijser [14] and the value of $C_n$ was improved to $C_n = 2^{\frac{n-1}{2}}$. Also, H. Quéffelec [18], A. Defant and P. Sevilla-Peris [7] observed that $C_n = \left(\frac{2}{\sqrt{\pi}}\right)^{n-1}$ also works in (1.1).

In the recent years considerable effort related to the Bohnenblust–Hille inequality has been made (see [3, 6–8] and references therein) but, as it happens to Grothendieck's constant, there are still open questions regarding the precise value and behavior of the Bohnenblust–Hille constants. The questions related to the Bohnenblust–Hille constants, although up to now less investigated than Grothendieck's constant, seem at least as challenging as those related to Grothendieck's constant. Besides the estimation of the precise values of $C_n$, their asymptotic growth is also an open problem. In this note we shall be focusing on the asymptotic behavior of these constants.

In 2010, A. Defant, D. Popa and U. Schwarting [8] presented a new proof of the Bohnenblust–Hille Theorem (also valid for the real case) and by exploring this proof and estimates from [13] the second and third authors of this note obtained better estimates for $C_n$ [17]. However these estimates, for big values of $n$, were obtained recursively and a closed (non recursive) formula could not be easily obtained as well as the asymptotic growth of the constants.

In this short note we complement the results from [17] and provide a closed formula (non recursive) for these better estimates in the Bohnenblust–Hille inequality. We also determine the asymptotic behavior of these estimates, showing that if $C_{\mathbb{R},n}$ and $C_{\mathbb{C},n}$ denote (respectively) these constants for the real and complex Bohnenblust–Hille inequality then:

(1) For every even positive integer $n$,

$$C_{\mathbb{R},n} = \left( \frac{\sqrt{\pi}}{\sqrt{2}} \right) C_{\mathbb{C},n} = 2^{\frac{n+2}{8}} \cdot r_n, \tag{1.2}$$

for certain sequence $\{r_n\}_n$, which we estimate numerically to belong to the interval $(1, 3/2)$.

(2) If $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then

$$\lim_{n\to\infty} \frac{C_{\mathbb{K},n}}{C_{\mathbb{K},n-1}} = 2^{\frac{1}{8}} . \lim_{n\to\infty} \frac{r_n}{r_{n-1}}.$$

In particular, if $\{r_n\}_{n\in\mathbb{N}}$ is in fact convergent (as our numerical estimates indicate), then

$$\lim_{n\to\infty} \frac{C_{\mathbb{K},n}}{C_{\mathbb{K},n-1}} = 2^{\frac{1}{8}}.$$

(3) It worths to be mentioned that for complex scalars (1.2) can be replaced by smaller constants:

$$C_{\mathbb{C},n} = \frac{2^{\frac{n}{8} + \frac{67}{n} + \frac{3}{4}}}{\pi^{\frac{36}{n} + \frac{1}{2}}} \cdot \left(3.9296 \times 10^{-3}\right)^{1/n} r_n$$

## 2. Bohnenblust–Hille constants: The real case

The following result appears in [17], as a consequence of results from [8]:

**Theorem 2.1.** *For every positive integer $n$ and real Banach spaces $X_1, \dots, X_n$,*

$$\Pi_{(\frac{2n}{n+1};1)}(X_1, \dots, X_n; \mathbb{R}) = \mathcal{L}(X_1, \dots, X_n; \mathbb{R}) \text{ and } \|.\|_{\pi(\frac{2n}{n+1};1)} \leq C_{\mathbb{R},n} \|.\|$$

*with*

$$C_{\mathbb{R},2} = 2^{\frac{1}{2}} \text{ and } C_{\mathbb{R},3} = 2^{\frac{5}{6}},$$

$$C_{\mathbb{R},n} = 2^{\frac{1}{2}} \left( \frac{C_{\mathbb{R},n-2}}{A^2_{\frac{2n-4}{n-1}}} \right)^{\frac{n-2}{n}} \text{ for } n > 3. \tag{2.1}$$

*In particular, if* $2 \leq n \leq 14$,

$$C_{\mathbb{R},n} = 2^{\frac{n^2+6n-8}{8n}} \text{ if } n \text{ is even, and} \tag{2.2}$$

$$C_{\mathbb{R},n} = 2^{\frac{n^2+6n-7}{8n}} \text{ if } n \text{ is odd.} \tag{2.3}$$

The above result and the next remark will be crucial for the main results in this note.

**Remark 2.2.** *Throughout this paper the sequence* $\{r_n\}_{n \in 2\mathbb{N}}$ *given by*

$$r_n = \frac{1}{2^{\frac{n-2}{4}} \cdot \left[ \prod_{k=1}^{\frac{n-2}{2}} \left( \frac{\Gamma(\frac{6k+1}{4k+2})}{\sqrt{\pi}} \right)^{2k+1} \right]^{1/n}} = \frac{\pi^{\frac{n^2-4}{8n}}}{2^{\frac{n-2}{4}} \cdot \left[ \prod_{k=1}^{\frac{n-2}{2}} \left( \Gamma(\frac{6k+1}{4k+2}) \right)^{2k+1} \right]^{1/n}}$$

*shall appear very often. Although it is not proved here, we have strong numerical evidence supporting the fact that the above sequence is convergent and, moreover,*

$$r_n \approx 1.44.$$

*The interested reader can see here below a table with some of the values of* $r_n$, *for* $n$ *even.*

| $n$ | $r_n$ |
|---:|---|
| *10* | *1.28682* |
| *30* | *1.37516* |
| *50* | *1.39747* |
| *100* | *1.41640* |
| *250* | *1.42943* |
| *500* | *1.43437* |
| *1,000* | *1.43707* |
| *5,000* | *1.43951* |
| *10,000* | *1.43986* |
| *15,000* | *1.43998* |
| *25,000* | *1.44008* |
| *40,000* | *1.44014* |
| *100,000* | *1.44021* |
| *300,000* | *1.44023* |
| *1,000,000* | *1.44025* |

As we have mentioned before, Theorem 2.1 does not furnish a closed formula for the constants of Bohnenblust–Hille inequality for big values of $n$. The estimates are recursive and makes difficult a complete comprehension of their growth. Our first result is a closed formula for $C_{\mathbb{R},n}$ with $n$ even:

**Theorem 2.3.** *If* $n$ *is an even positive integer, then*

$$C_{\mathbb{R},n} = 2^{\frac{n+2}{8}} r_n.$$

*Proof.* Let us begin by noticing that

$$
\begin{aligned}
C_{\mathbb{R},4} &= \sqrt{2}\left(\frac{C_{\mathbb{R},2}}{A^2_{\frac{4}{3}}}\right)^{\frac{2}{4}},\\
C_{\mathbb{R},6} &= \sqrt{2}\left[\frac{\sqrt{2}\left(\frac{C_{\mathbb{R},2}}{A^2_{\frac{4}{3}}}\right)^{\frac{2}{4}}}{A^2_{\frac{8}{5}}}\right]^{\frac{4}{6}} = \frac{\left(2^{\frac{1}{2}+\frac{1}{2}\cdot\frac{4}{6}}\right)(C_{\mathbb{R},2})^{\frac{2}{4}\cdot\frac{4}{6}}}{\left(A^2_{\frac{4}{3}}\right)^{\frac{2}{4}\cdot\frac{4}{6}}\left(A^2_{\frac{8}{5}}\right)^{\frac{4}{6}}},\\
C_{\mathbb{R},8} &= \sqrt{2}\left[\frac{\frac{\left(2^{\frac{1}{2}+\frac{1}{2}\cdot\frac{4}{6}}\right)(C_{\mathbb{R},2})^{\frac{2}{4}\cdot\frac{4}{6}}}{\left(A^2_{\frac{4}{3}}\right)^{\frac{2}{4}\cdot\frac{4}{6}}\left(A^2_{\frac{8}{5}}\right)^{\frac{4}{6}}}}{A^2_{\frac{12}{7}}}\right]^{\frac{6}{8}} = \frac{\left(2^{\frac{1}{2}+\left(\frac{1}{2}+\frac{1}{2}\cdot\frac{4}{6}\right)\cdot\frac{6}{8}}\right)(C_{\mathbb{R},2})^{\frac{2}{4}\cdot\frac{4}{6}\cdot\frac{6}{8}}}{\left(A^2_{\frac{4}{3}}\right)^{\frac{2}{4}\cdot\frac{4}{6}\cdot\frac{6}{8}}\left(A^2_{\frac{8}{5}}\right)^{\frac{4}{6}\cdot\frac{6}{8}}\left(A^2_{\frac{12}{7}}\right)^{\frac{6}{8}}},
\end{aligned}
$$

and so on.

Now, using the fact that

$$A_p = \sqrt{2}\left(\frac{\Gamma((p+1)/2)}{\sqrt{\pi}}\right)^{1/p},$$

we can define

$$
\begin{aligned}
s_n &= \left(A^2_{\frac{4}{3}}\right)^{\frac{2}{4}\cdot\frac{4}{6}\cdots\frac{n-2}{n}}\left(A^2_{\frac{8}{5}}\right)^{\frac{4}{6}\cdot\frac{6}{8}\cdots\frac{n-2}{n}}\left(A^2_{\frac{12}{7}}\right)^{\frac{6}{8}\cdot\frac{8}{10}\cdots\frac{n-2}{n}}\cdot\cdots\cdot\left(A^2_{\frac{2n-4}{n-1}}\right)^{\frac{n-2}{n}} =\\
&= 2^{\frac{n-2}{4}}\cdot\left[\prod_{k=1}^{\frac{n-2}{2}}\left(\frac{\Gamma(\frac{6k+1}{4k+2})}{\sqrt{\pi}}\right)^{2k+1}\right]^{1/n} = \frac{2^{\frac{n-2}{4}}\cdot\left[\prod_{k=1}^{\frac{n-2}{2}}\left(\Gamma(\frac{6k+1}{4k+2})\right)^{2k+1}\right]^{1/n}}{\pi^{\frac{n^2-4}{8n}}}
\end{aligned}
$$

Notice that (see Remark 2.2) $r_n = 1/s_n$. It can be easily checked that

$$C_{\mathbb{R},n} = 2^{\frac{n+2}{8}}\cdot r_n. \tag{2.4}$$

Indeed, it suffices with noticing that $C_{\mathbb{R},2}=\sqrt{2}$ and that

$$
\begin{aligned}
C_{\mathbb{R},n}/r_n &= 2^{\frac{1}{2}+\frac{1}{2}\cdot\frac{(n-2)}{n}+\frac{1}{2}\cdot\frac{(n-4)}{n}+\frac{1}{2}\cdot\frac{(n-6)}{n}+\cdots+\frac{1}{2}\cdot\frac{(n-(n-4))}{n}}.\sqrt{2}^{\frac{2}{n}}\\
&= 2^{\left(\frac{1}{2}+\frac{1}{2n}[(n-2)+\cdots+4]\right)+\frac{1}{n}}\\
&= 2^{\frac{n+2}{8}}.
\end{aligned}
$$

□

**Remark 2.4.** *Let us note that, although the previous theorem was proved for $n$ even, a similar (but not identical) result holds for all $n\in\mathbb{N}$. On the other hand, the asymptotic behavior of the constants for $n$ odd remains identical. The same also holds for Theorem 3.2.*

Some values of the sequence $\{C_{\mathbb{R},n}\}_n$ (using the above formula for $C_{\mathbb{R},n}$) are shown in the following table.

| $n$ | $C_{\mathbb{R},n}$ |
|---|---|
| 30 | $\approx 22$ |
| 50 | $\approx 126$ |
| 100 | $\approx 9757$ |
| 500 | $\approx 10^{19}$ |
| 1,000 | $\approx 10^{37}$ |
| 5,000 | $\approx 10^{188}$ |

The following result will give us the behavior of $\frac{C_{\mathbb{R},n}}{C_{\mathbb{R},n-1}}$.

**Theorem 2.5.** *For the real case,*

$$\lim_{n\to\infty}\frac{C_{\mathbb{R},n}}{C_{\mathbb{R},n-2}}=2^{\frac{1}{4}}.\lim_{n\to\infty}\frac{r_n}{r_{n-2}}.$$

*In particular, assuming the convergence of* $\{r_n\}$,

$$\lim_{n\to\infty}\frac{C_{\mathbb{R},n}}{C_{\mathbb{R},n-1}}=2^{\frac{1}{8}}.$$

*Proof.* We can work for even and odd integers since the asymptotic behavior for both cases is the same. The first estimate is clear, since

$$\frac{C_{\mathbb{R},n}}{C_{\mathbb{R},n-2}}=\frac{2^{\frac{n+2}{8}}r_n}{2^{\frac{n}{8}}r_{n-2}}=2^{\frac{1}{4}}\cdot\frac{r_n}{r_{n-2}}.$$

If $\{r_n\}$ is convergent we have

$$2^{\frac{1}{4}}=\lim_{n\to\infty}\frac{C_{\mathbb{R},n}}{C_{\mathbb{R},n-2}}=\lim_{n\to\infty}\left(\frac{C_{\mathbb{R},n}}{C_{\mathbb{R},n-1}}.\frac{C_{\mathbb{R},n-1}}{C_{\mathbb{R},n-2}}\right)=\lim_{n\to\infty}\left(\frac{C_{\mathbb{R},n}}{C_{\mathbb{R},n-1}}\right)^2$$

and thus,

$$\lim_{n\to\infty}\frac{C_{\mathbb{R},n}}{C_{\mathbb{R},n-1}}=2^{\frac{1}{8}}.$$

□

## 3. Bohnenblust–Hille constants: The complex case

The version of Theorem 2.1 for complex scalars is:

**Theorem 3.1.** *For every positive integer* $m$ *and complex Banach spaces* $X_1,\dots,X_m$,

$$\Pi_{(\frac{2m}{m+1};1)}(X_1,...,X_m;\mathbb{C})=\mathcal{L}(X_1,...,X_m;\mathbb{C})\ \text{and}\ \|.\|_{\pi(\frac{2m}{m+1};1)}\le C_{\mathbb{C},m}\|.\|$$

*with*

$$C_{\mathbb{C},m}=\left(\frac{2}{\sqrt{\pi}}\right)^{m-1}\quad \text{for } m=2,3,$$

$$C_{\mathbb{C},m}\le\frac{2^{\frac{m+2}{2m}}}{\pi^{1/m}}\left(\frac{1}{A^2_{\frac{2m-4}{m-1}}}\right)^{\frac{m-2}{m}}(C_{\mathbb{C},m-2})^{\frac{m-2}{m}}\quad \text{for } m\ge 4.$$

*In particular, if* $4\le m\le 14$ *we have*

$$C_{\mathbb{C},m}\le\left(\frac{1}{\pi^{1/m}}\right)2^{\frac{m+4}{2m}}(C_{\mathbb{C},m-2})^{\frac{m-2}{m}}.$$

Following [17, Theorem 3.2], there exists a sequence $\{B_n\}_{n\in\mathbb{N}}$ such that, for every $m\in\mathbb{N}$,

$$\frac{C_{\mathbb{C},n}}{C_{\mathbb{C},n-2}^{\frac{n-2}{n}}}\quad=\quad B_n$$

where

$$B_n=\frac{2^{(n+2)/(2n)}}{\pi^{1/n}}\cdot\left(\frac{1}{A^2_{\frac{2n-4}{n-1}}}\right)^{\frac{n-2}{n}}.$$

Now, making some algebraic manipulations, and keeping in mind that

$$A_p=\sqrt{2}\left(\frac{\Gamma((p+1)/2)}{\sqrt{\pi}}\right)^{1/p},$$

we have

$$B_n\quad=\quad\left(\frac{\pi}{2}\right)^{(n-3)/(2n)}\cdot 2^{3/(2n)}\cdot\frac{1}{\Gamma\left(\frac{3n-5}{2n-2}\right)^{(n-1)/n}}$$

Now, from the continuity of the Gamma function, together with considering equivalent infinities, one has that

$$\lim_{n\to\infty} B_n = \lim_{n\to\infty} \frac{\left(\frac{\pi}{2}\right)^{(n-3)/(2n)} \cdot 2^{3/(2n)}}{\Gamma\left(\frac{3n-5}{2n-2}\right)^{(n-1)/n}} = \frac{\left(\frac{\pi}{2}\right)^{1/2} \cdot 2^0}{\Gamma(3/2)} = \sqrt{2},$$

using the known fact that $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$. Thus, we have shown that

$$\lim_{n\to\infty} \left( \frac{C_{\mathbb{C},n}}{C_{\mathbb{C},n-2}^{\frac{n-2}{n}}} \right) = \sqrt{2}.$$

Our aim now shall be to find a closed expression for the values of $C_{\mathbb{C},m}$ in order to be able to study the asymptotic behavior.

**Theorem 3.2.** *If $n$ is an even positive integer, then*

$$C_{\mathbb{C},n} = 2^{\frac{n+2}{8}} \cdot \frac{\sqrt{2}}{\sqrt{\pi}} \cdot r_n.$$

The proof is very similar to that of Theorem 2.3 and we spare the details of it to the interested reader. The case $n$ odd has a very similar formula. Next, the following result, of identical proof as in Theorem 2.5, is now due:

**Theorem 3.3.** *For the complex case,*

$$\lim_{n\to\infty} \frac{C_{\mathbb{C},n}}{C_{\mathbb{C},n-2}} = 2^{\frac{1}{4}} . \lim_{n\to\infty} \frac{r_n}{r_{n-2}}.$$

*In particular, assuming the convergence of $\{r_n\}$,*

$$\lim_{n\to\infty} \frac{C_{\mathbb{C},n}}{C_{\mathbb{C},n-1}} = 2^{\frac{1}{8}}.$$

3.1. **Some remarks.** The following result was also obtained in [17] as a consequence of results from [8], providing smaller constants for the complex case:

**Theorem 3.4.** [17, Theorem 3.2] *For every positive integer $n$ and every complex Banach spaces $X_1, \ldots, X_n$,*

$$\Pi_{\left(\frac{2n}{n+1};1\right)}(X_1, \ldots, X_n; \mathbb{C}) = \mathcal{L}(X_1, \ldots, X_n; \mathbb{C}) \text{ and } \|.\|_{\pi\left(\frac{2n}{n+1};1\right)} \leq C_{\mathbb{C},n} \|.\|$$

*with*

$$C_{\mathbb{C},n} = \left(\frac{2}{\sqrt{\pi}}\right)^{n-1} \text{ for } m = 2, 3, 4, 5, 6, 7,$$

$$C_{\mathbb{C},n} \leq \frac{2^{\frac{n+2}{2n}}}{\pi^{1/n}} \left(\frac{1}{A^2_{\frac{2n-4}{n-1}}}\right)^{\frac{n-2}{n}} (C_{\mathbb{C},n-2})^{\frac{n-2}{n}} \text{ for } n \geq 8.$$

*In particular, for $8 \leq n \leq 14$ we have*

$$C_{\mathbb{C},n} \leq \left(\frac{1}{\pi^{1/n}}\right) 2^{\frac{n+4}{2n}} (C_{\mathbb{C},n-2})^{\frac{n-2}{n}}.$$

By using the constants above we can obtain a closed formula with smaller constants for the complex case. But, these new constants have the same asymptotic behavior of the previous. It can be checked that

$$C_{\mathbb{C},14} = \frac{2^{30/7}}{\pi^{19/14}}.$$

In a similar fashion as the calculations we made for the real case in Theorem 2.3, it can be seen (we spare the details to the reader) that, for even values of $n \geq 16$:

$$C_{\mathbb{C},n} = \frac{2^{\frac{n}{8}+\frac{67}{n}+\frac{3}{4}}}{\pi^{\frac{36}{n}+\frac{1}{2}}} \cdot \left(\prod_{k=1}^{6} \Gamma\left(\frac{6k+1}{4k+2}\right)^{2k+1}\right)^{1/n} r_n. \tag{3.1}$$

Evaluating $\prod_{k=1}^{6}\Gamma\left(\frac{6k+1}{4k+2}\right)^{2k+1}$ we obtain

$$C_{\mathbb{C},n}=\frac{2^{\frac{n}{8}+\frac{67}{n}+\frac{3}{4}}}{\pi^{\frac{36}{n}+\frac{1}{2}}}\cdot\sqrt[n]{0.003929571803}\cdot r_n$$

Of course, a similar procedure can be made for odd values of $n$.

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## References


[1] N. Alon, K. Makarychev, Y. Makarychev, and A. Naor, *Quadratic forms on graphs*, Inventiones Math. **163** (2006), 499–522.

[2] M. Braverman, K. Makarychev, Y. Makarychev, and A. Naor, *The Grothendiek constant is strictly smaller than Krivine's bound*, arXiv:1103.6161v2 (2011).

[3] G. Botelho, H.-A. Braunss, H. Junek, and D. Pellegrino, *Inclusions and coincidences for multiple summing multilinear mappings*, Proc. Amer. Math. Soc. **137** (2009), 991–1000.

[4] H. F. Bohnenblust and E. Hille, *On the absolute convergence of Dirichlet series*, Ann. of Math. (2) **32** (1931), 600–622.

[5] A. M. Davie, *Quotient algebras of uniform algebras*, J. London Math. Soc. (2) **7** (1973), 31–40.

[6] A. Defant, L. Frerick, J. Ortega-Cerdà, M. Ounaïes, and K. Seip, *The Bohnenblust–Hille inequality for homogeneous polynomials is hypercontractive*, Ann. of Math. (2) **174** (2011), 485–497.

[7] A. Defant and P. Sevilla-Peris, *A new multilinear insight on Littlewood's 4/3-inequality*, J. Funct. Anal. **256** (2009), 1642–1664.

[8] A. Defant, D. Popa, and U. Schwarting, *Coordinatewise multiple summing operators in Banach spaces*, J. Funct. Anal. **259** (2010), 220–242.

[9] J. Diestel, J. Fourie, and J. Swart, *The metric theory of tensor products*, American Mathematical Society (2008).

[10] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators*, Cambridge Studies in Advanced Mathematics (1995).

[11] P. C. Fishburn and J. A. Reeds, *Bell inequalities, Grothendieck's constant, and root two*, SIAM J. Disc. Math. **7** (1994), 48–56.

[12] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. Sao Paulo **8** (1953), 1–79.

[13] U. Haagerup, *The best constants in the Khintchine inequality*, Studia Math. **70** (1982), 231–283.

[14] S. Kaijser, *Some results in the metric theory of tensor products*, Studia Math. **63** (1978), 157–170.

[15] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in $L_p$ spaces and their applications*, Studia Math. **29** (1968), 276–326.

[16] J. E. Littlewood, *On bounded bilinear forms in an infinite number of variables*, Q. J. Math. **1** (1930), 164–174.

[17] D. Pellegrino and J. B. Seoane-Sepúlveda, *Improving the constants for real and complex Bohnenblust–Hille inequality*, Preprint, arXiv 1010.0461v2, October 2010.

[18] H. Queffélec, *H. Bohr's vision of ordinary Dirichlet series; old and new results*, J. Anal. **3** (1995), 43–60.



Departamento de Análisis Matemático,
Facultad de Ciencias Matemáticas,
Plaza de Ciencias 3,
Universidad Complutense de Madrid,
Madrid, 28040, Spain.
*E-mail address*: gustavo_fernandez@mat.ucm.es

Departamento de Matemática,
Universidade Federal da Paraíba,
58.051-900 - João Pessoa, Brazil.
*E-mail address*: pellegrino@pq.cnpq.br

Departamento de Análisis Matemático,
Facultad de Ciencias Matemáticas,
Plaza de Ciencias 3,
Universidad Complutense de Madrid,
Madrid, 28040, Spain.
*E-mail address*: jseoane@mat.ucm.es